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Abstract

Let $\Omega$ be a regular triangulation of a two dimensional domain and $S_n^\delta(\Omega)$ be a vector space of functions in $C^r$ whose restriction to each small triangle in $\Omega$ is a polynomial of total degree at most $n$. Dimensions of bivariate spline spaces $S_n^\delta(\Omega)$ over a special kind of triangulation, called the unconstricted triangulation, were given by Farin in the paper [J. Comput. Appl. Math. 192(2006), 320-327]. In this paper, a counter example is given to show that the condition used in the main theorem in Farin’s paper is not correct, and then an improved necessary and sufficient condition is presented.

1. A counter example

In [2], by introducing two kinds of construction operations, called a flap and a pair of triangles respectively, the so-called unconstricted triangulation was first defined, which can be obtained by recursively adding a flap or a pair of triangles to a subtriangulation started from a single triangle. In Section 3 in [2], the dimension of $S_3^\delta(\Omega)$ over the unconstricted triangulation was given. Then in Section 4, the construction of a minimal determining set for the spline space $S_n^\delta(\Omega)$ over a star $\nu^\star$ was further considered. And finally in Section 5, the dimension of the spline space $S_n^\delta(\Omega)$ over the unconstricted triangulation $\Omega$ was determined by recursively using the results over stars presented in Section 4.

For a star $\nu^\star$ which is obtained from $\nu^\star-2$ by adding a pair of triangles, $\delta^e_\nu(b)$ was defined in [2] as

$$\delta^e_\nu(b) = \dim S_n^\delta(\nu^\star) - \dim S_n^\delta(\nu^\star-2),$$

where $b$ is the valence of an interior vertex $\nu$.

The key step in the proof of the theorem in Section 5 in [2] is based on the statement “if $\delta^e_\nu(b) \geq 0$ then a minimal determining set for $S_n^\delta(\nu^\star)$ can be obtained by adding some other Bézier ordinates to the minimal determining set for $S_n^\delta(\nu^\star-2)$”. However, it is found in this section that this statement is not always true. Here is a counter example.

Counter example. Let us take $b = 5$, $n = 5$ and $r = 2$, and let $\nu^\star-2 = \Delta_1 \nu_2 \nu_3 \cup \Delta_2 \nu_2 \nu_3 \cup \Delta_3 \nu_2 \nu_4$ with $\angle_1 \nu_2 \nu_3 \in (\frac{\pi}{2}, \pi)$. The star $\nu^\star$ is obtained by adding a pair of triangles $\Delta_1 \nu_2 \nu_5 \cup \Delta_3 \nu_2 \nu_4$ to $\nu^\star-2$, where $\angle_2 \nu_2 \nu_5 \in (0, \frac{\pi}{2})$ and $\angle_1 \nu_2 \nu_4 \in (0, \frac{\pi}{2})$, as shown in Fig. 1.

We first consider the spline space $S_2^2(\nu^\star-2)$. It follows from [4], [5] that $\dim S_2^2(\nu^\star-2) = 33$. And a minimal determining set for $S_3^2(\nu^\star-2)$ can be easily chosen as the Bézier ordinates with respect to the domain points marked by “•” as shown in Fig. 1(a), which is denoted by $P_3^3(\nu^\star-2)$.

Then we consider the spline space $S_2^2(\nu^\star)$. It follows from [3] or [4] that $\dim S_2^2(\nu^\star) = 36$. Thus

$$\delta^e_\nu(5) = \dim S_2^2(\nu^\star) - \dim S_2^2(\nu^\star-2) = 36 - 33 > 0.$$  (2)

However, for this case, we can show that any minimal determining set for $S_2^2(\nu^\star)$, denoted by $P_3^3(\nu^\star)$, cannot be obtained by adding some Bézier ordinates to the minimal determining set $P_3^3(\nu^\star-2)$.

In fact, for the space $S_2^2(\nu^\star)$, let us consider $C^2$ smoothness conditions in $D_3(\nu)$, where $D_3(\nu)$ is the third disk around the vertex $\nu$, as shown in Fig. 1(b). Let

$$\nu_4 = \alpha_1 \nu + \beta_1 \nu_1 + \gamma_1 \nu_5,$$  (3)

$$\nu_5 = \alpha_2 \nu + \beta_2 \nu_3 + \gamma_2 \nu_4.$$  (4)
and π, β, γ and δ denote the corresponding Bézier ordinates of s ∈ S^2(v^*) with respect to four domain points A, B, C and D, respectively. If we assume that all the Bézier ordinates with respect to all domain points marked by “•” in D_b(v) in Fig. 1(b) vanish, then it follows from C^2 smoothness conditions [1] that

\[
\begin{pmatrix}
γ_1 & -1 & 0 & 0 \\
γ_2 & 0 & -1 & 0 \\
0 & 0 & -1 & γ_2 \\
0 & -1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
π \\
β \\
γ \\
δ
\end{pmatrix}
= 0.
\] (5)

Because of the assumption in [2] that the triangulation does not contain any degenerated (or called singular) edge, we have γ_1 = S_{∂v}^{∂v_1}v_4 ≠ 0, γ_2 = S_{∂v}^{∂v_2}v_3 ≠ 0. In addition,

\[
g_1γ_2 = \frac{S_{∂v}^{∂v_1}v_4}{S_{∂v}^{∂v_2}v_3} = \frac{\sin{∠v_1v_4}}{\sin{∠v_2v_4}}, \quad \frac{S_{∂v}^{∂v_2}v_3}{S_{∂v}^{∂v_1}v_4} = \frac{\sin{∠v_2v_3}}{\sin{∠v_1v_3}}.
\] (6)

It is noted that ∠v_1v_3 ∈ (π, π), so π − ∠v_1v_4 < ∠v_1v_5 < \frac{2π}{3}, which together with the assumptions ∠v_1v_3 ∈ (0, π) and ∠v_3v_4 ∈ (0, π) yield that

\[
\frac{π}{2} < π − ∠v_1v_4 < ∠v_1v_5 < π.
\] (7)

\[
\frac{π}{2} < π − ∠v_1v_3 < ∠v_1v_5 < π.
\] (8)

Since the function sin x decreases monotonously in the interval (π, π), we have

\[
\sin{∠v_1v_4} < \sin(π − ∠v_1v_4) = \sin{∠v_1v_4},
\] (9)

\[
\sin{∠v_1v_5} < \sin(π − ∠v_1v_5) = \sin{∠v_1v_5}.
\] (10)

So we have γ_1γ_2 < 1, and thus

\[
det \begin{pmatrix}
γ_1 & -1 & 0 & 0 \\
γ_2 & 0 & -1 & 0 \\
0 & 0 & -1 & γ_2 \\
0 & -1 & 0 & 0
\end{pmatrix} = γ_1γ_2(1 − γ_1γ_2) ≠ 0.
\] (11)

This means that π = β = γ = δ = 0. Therefore the Bézier ordinate with respect to domain point D marked by “•” must be excluded from the minimal determining set P^2_β(v^*) for S^2_b(v^*) though it is in P^2_β(v^*). A correct minimal determining set for S^2_β(v^*) with respect to domain points marked by “•” is displayed in Fig. 1(b).

The counter example reveals that the condition δ^2_n(b) ≥ 0 is not a sufficient condition to guarantee that there exists a minimal determining set for S^2_n(v^*) which can be obtained by adding some Bézier ordinates to a minimal determining set for S^2_n(v^*). In the next section, we shall give a necessary and sufficient condition.

2. A necessary and sufficient condition

Let P^r_n(v^*) be a minimal determining set for S^r_n(v^*) which consists of Bézier ordinates with respect to domain points taken ring by ring from R_0(v) to R_n(v), where R_i(v) is the i-th ring around the vertex v in the star v^*. We have the following

**Lemma 1.** Let N^r_n(b, i) = |P^r_n(v^*) ∩ R_i(v)|, i = 0, 1, ..., n, where |·| denotes the cardinality of the set. Then

\[
N^r_n(b, i) = \begin{cases}
0 < i ≤ r, & \frac{i + 1}{2} \left( r + 2 \right) + \sum_{j=1}^{i-r} (r + j + 1 − je), & r < i ≤ n,
\end{cases}
\] (12)

where e is the number of edges with different slopes attached to the vertex v.

**Proof.** When 0 ≤ i ≤ r, it is well-known that

\[
dim S^r_n(v^*) = \left( \frac{i + 2}{2} \right).
\] (13)

When r < i ≤ n, it follows from [3] or [4] that

\[
dim S^r_n(v^*) = \left( \frac{r + 2}{2} \right) + \left( \frac{i − r + 1}{2} \right) b + \sum_{j=1}^{i-r} (r + j + 1 − je).
\] (14)

Thus,

\[
N^r_n(b, i) = \dim S^r_n(v^*) − \dim S^r_{i-1}(v^*) = \begin{cases}
1, & \text{if } 0 ≤ i ≤ r, \\
\frac{i + 1}{2}, & \text{if } r < i ≤ n.
\end{cases}
\]

Similarly, let P^r_n(v^*-2) be a minimal determining set for S^r_n(v^*-2) which consists of Bézier ordinates with respect to domain points taken ring by ring from R_0(v) to R_n(v), where R_i(v) is the i-th ring around the vertex v in the triangulation v^*-2. We also have

**Lemma 2.** Let N^r_n(b, i) = |P^r_n(v^*-2) ∩ R_i(v)|, i = 0, 1, ..., n. Then

\[
N^r_n(b, i) = \begin{cases}
0 < i ≤ r, & \left( \frac{i + 1}{2} \right)(b − 3), \quad r < i ≤ n.
\end{cases}
\] (15)

**Proof.** When 0 ≤ i ≤ r, it is well-known that

\[
dim S^r_n(v^*-2) = \left( \frac{i + 2}{2} \right).
\] (16)

When r < i ≤ n, it follows from the dimensional formula for cross-cut partition given by [4], [5] that

\[
dim S^r_n(v^*-2) = \left( \frac{i + 2}{2} \right) + \left( \frac{i − r + 1}{2} \right)(b − 3).
\] (17)
where \( b - 3 \) is the number of the cross-cut edges. Thus we have
\[
\underline{N}_n(b, i) = \dim S^r_n(v^{r - 2}) - \dim S^r_{n-1}(v^{r - 2})
\]
\[
= \begin{cases} 
1, & i = 0, \\
i + 1, & 1 \leq i \leq r, \\
r + b - 1, & i = r + 1, \\
i + 1 + (i-r)(b-3), & r + 1 < i \leq n
\end{cases}
\]
where \( \dim S^r_{n-1}(v^{r - 2}) = 0 \). The proof of the lemma is completed.

Based on Lemma 1 and 2, we have the following

**Theorem.** Let \( \Omega \) be an unconstricted triangulation with nonsingular vertices, that is, triangulations without vertices with collinear edges emanating from them. And let \( A^*_n(b, i) = N^*_n(b, i) - \underline{N}_n(b, i) \), \( i = 0, 1, \ldots, n \). A necessary and sufficient condition for existing a pair of minimal determining sets \( P^*_n(v^{r - 2}) \) for \( S^r_n(v^r) \) and \( P^*_n(v^r) \) for \( S^r_n(v^r) \) to satisfy \( P^*_n(v^{r - 2}) \subseteq P^*_n(v^r) \) is
\[
A^*_n(b, i) = \begin{cases} 
0, & 0 \leq i \leq r, \\
2i - 3r - 1 + (i + 1 - (i-r)e) +, & r < i \leq n.
\end{cases}
\]  
(18)

**Proof.** Suppose that a pair of minimal determining sets \( P^*_n(v^{r - 2}) \) for \( S^r_n(v^{r - 2}) \) and \( P^*_n(v^r) \) for \( S^r_n(v^r) \) exist to satisfy \( P^*_n(v^{r - 2}) \subseteq P^*_n(v^r) \). Then
\[
N^*_n(b, i) \geq \underline{N}_n(b, i), 
\]
i.e.,
\[
A^*_n(b, i) \geq 0, 
\]
i.e., \( i = 0, 1, \ldots, n \).

Thus, the inequality (18) holds.

Now suppose that the inequality (18) holds. If we take \( i = r + 1 \), the inequality (18) becomes
\[
A^*_n(b, r + 1) = 1 - r + (r + 2 - e)_+ \geq 0.
\]

Specifically,
\[
1 - r \geq 0, \quad \text{when } r + 2 - e \leq 0,
\]
or
\[
3 - e \geq 0, \quad \text{when } r + 2 - e \geq 0.
\]

That is to say, if we suppose that the inequality (18) holds, then there exist at least five possibilities as follows

- **Case 1)** \( r = 0 \), when \( e = 2 \),
- **Case 2)** \( r = 0 \), when \( e \geq 3 \),
- **Case 3)** \( r = 1 \), when \( e \geq 3 \),
- **Case 4)** \( e = 3 \), when \( r \geq 1 \),
- **Case 5)** \( e = 2 \), when \( r \geq 2 \).

Since \( \Omega \) does not contain any degenerated edge, both Case 1) and Case 5) can be discarded. In addition, the proof for Case 2) is trivial, we only consider Case 3) and Case 4).

For Case 3), if \( n = r = 1 \), then both spline spaces \( S^r_n(v^{r - 2}) \) and \( S^r_n(v^r) \) are degenerated into the polynomial space \( P_1 \), thus the minimal determining set \( P^*_n(v^{r - 2}) \) can take to be the same to \( P^*_n(v^{r - 2}) \), the conclusion holds.

If \( n > r = 1 \), as shown in Fig. 2, when we add a pair of triangles \( \Delta v_1v_2v_3 \cup \Delta v_1v_2v_4 \) to \( v^{r - 2} \), by using the \( C^1 \) smoothness conditions along the two edges \( v_1v_2 \) and \( v_2v_4 \), all the Bézier ordinates associated with domain points \( e \) in \( \Delta v_1v_2v_3 \cup \Delta v_2v_4v_5 \) with distance to two edges \( v_1v_2 \) and \( v_2v_4 \) being 1 can be determined by \( P^*_n(v^{r - 2}) \). Further, since there is no degenerated edge in the triangulation \( \Omega \), the Bézier ordinate associated with domain point \( A \) can be also determined by \( P^*_n(v^{r - 2}) \). Next, by using the \( C^1 \) smoothness conditions along the edge \( v_3v_5 \), we can obtain other \( n - 2 \) equations. Therefore the total number of independent Bézier ordinates in \( \Delta v_1v_2v_3 \cup \Delta v_2v_4v_5 \) is at least
\[
((n - 1)^2 - 1) - (n - 2) = (n - 1)(n - 2), \quad (19)
\]
which is nonnegative as \( n \geq r = 1 \). So we can construct a minimal determining set \( P^*_n(v^r) \) for \( S^r_n(v^r) \) which contains \( P^*_n(v^{r - 2}) \) as its subset, i.e., \( P^*_n(v^{r - 2}) \subseteq P^*_n(v^r) \).

Now consider Case 4). Since the unconstricted triangulation \( \Omega \) does not contain any vertex with collinear edges emanating from it, it follows from \( e = 3 \) that \( b = 3 \), i.e., \( b = 3 \) and \( r \geq 1 \).

In this case, the star \( v^r \) is formed by adding a pair of triangles \( \Delta v_1v_2v_3 \cup \Delta v_2v_4v_5 \) to \( v^{r - 2} = \Delta v_1v_2v_3 \), see Fig. 3. On one hand, a minimal determining set \( P^*_n(v^{r - 2}) \) for \( S^r_n(v^{r - 2}) \) can be taken all the Bézier ordinates associated with all \( \binom{n + 2}{2} \) domain points in \( \Delta v_1v_2v_3v_4v_5 \).

On the other hand, we have from [4], [5] that
\[
\dim S^r_n(v^r) = \left( n + \frac{2}{2} \right) + d_n(3), \quad (20)
\]
where
\[ d_r^n(3) = \left(n - r - \left\lfloor \frac{r+1}{2} \right\rfloor\right) + \left(n - 2r + \left\lfloor \frac{r+1}{2} \right\rfloor\right) \geq 0 \]
is the dimension of the solution space of the system consisting from the conformality conditions around the vertex \( v \) with \( \lfloor x \rfloor \) denoting the maximal integer which is not greater than \( x \), and \( \left(n + 2 \right)/2 \) is the dimension contributed by the bivariate polynomial of degree \( n \) in the source triangle \( \Delta v_1vv_2 \) which is exactly the cardinality of the minimal determining set \( P_r^n(v^*-2) \). Hence a minimal determining set \( P_r^n(v^*) \) for \( S_r^n(v^*) \) can be constructed by adding \( d_r^n(3) \) independent Bézier ordinates to \( P_r^n(v^*-2) \), i.e., \( P_r^n(v^*-2) \subseteq P_r^n(v^*) \).

The proof of the theorem is completed.  \( \Box \)

Furthermore, for \( 0 \leq m \leq n \), let us introduce \( \delta_r^m(b) = \dim S_r^m(v^*) - \dim S_r^{m-2}(v^*) \), then

\[
\delta_r^m(b) = \sum_{i=0}^{m} \left( \dim S_i^r(v^*) - \dim S_{i-1}^r(v^*) \right) - \sum_{i=0}^{m} \left( \dim S_i^r(v^*-2) - \dim S_{i-1}^r(v^*-2) \right)
= \sum_{i=0}^{m} \mathcal{N}_r^m(b, i) - \sum_{i=0}^{m} \mathcal{N}_r^m(b, i)
= \begin{cases} 
0, & m \leq r, \\
\sum_{i=r+1}^{m} A_r^m(b, i), & r < m \leq n.
\end{cases}
\]

Therefore, another necessary and sufficient condition equivalent to Eq.(18) is
\[
\delta_r^m(b) = \begin{cases} 
0, & 0 \leq m \leq r, \\
\geq 0, & r < m \leq n.
\end{cases}
\]

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